ON SOME CONTACT PROBLEMS FOR COMPOSITE HALF-SPACES

PMM Vol. 31, No. 6, 1967, pp. 1001-1008

B.L. ABRAMIAN and N.KH. ARUTIUNIAN
(Yerevan')

(*Received June 5, 1967*)

Solutions are presented herein of some contact problems connected with the torsion of a composite half-space. In the general case the problem of the torsion of a composite elastic half-space is examined by means of the rotation of a stiff finite cylinder welded into a vertical recess of this half-space. Moreover, the following particular problems on the torsion of such a half-space are considered.

1) A composite half-space with a vertical elastic infinite core, twisted by means of the rotation of a stiff stamp affixed to the upper endplate of the elastic core.

2) A half-space with a vertical cylindrical infinite hole, twisted by means of the rotation of a stiff finite cylinder welded into the upper part of this hole.

In the general case the solution of the problem reduces to the solution of an integral equation of the second kind on a half-line. The question of the solvability of this fundamental integral equation is investigated, and it is shown that its solution may be constructed by successive approximations.

Let us note that the problem of the torsion of a homogeneous half space and of an elastic layer by means of rotation of a stiff stamp has been considered by Rostovtsev [1], Reissner and Sagoci [2], Ufliand [3], Florence [4], Grilitskii [5] and others

The problem of the torsion of a circular cylindrical rod and the half-space welded to it which are subject to a torque applied to the free endface of the rod has been considered by Grilitskii and Kizyma [6].

The torsion of an elastic half-space with a vertical cylindrical inclusion of some other material by the rotation of a stiff stamp on the surface of this half-space has been considered in [7], wherein it has been assumed that the stamp is symmetrically disposed relative to the axis of the inclusion and lies simultaneously on both materials.

1. Torsion of a composite elastic half-space by rotation of a stiff finite cylinder welded into a vertical recess of this half-space. Let us consider the problem of torsion of a composite half-space with a vertical elastic cylindrical core located below the surface of the half-space by its upper endface (Fig. 1). A stiff cylinder of the same radius as the core is welded into the recess of the half-space, and torsion of the half-space is accomplished by rotating this stiff cylinder or stamp. To solve the problem, we direct the z-axis along the core axis and we divide the axial section of the half-space with core into three sub-domains $D_i (i = 1, 2, 3)$.

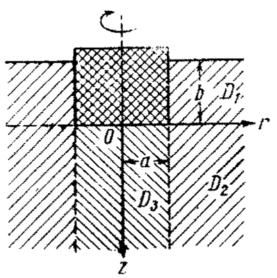

Fig. 1

We seek the displacement function $\Psi(r, z)$ which satisfies the differential Eq.

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{3}{r}\frac{\partial\Psi}{\partial r} + \frac{\partial^2\Psi}{\partial z^2} = 0 \qquad (1.1)$$

in the form

$$\Psi(r, z) = \Psi_i(r, z) \qquad \text{в } D_i\,(i, 2, 3) \qquad (1.2)$$

$$\Psi_1(r, z) = \frac{1}{r}\int_0^\infty \xi\,[D(\xi)\operatorname{sh}\xi z + F(\xi)\operatorname{ch}\xi z]\,W_1(\xi r)\,d\xi + \tag{1.3}$$

$$+\frac{1}{r}\sum_{k=1}^{\infty} H_k K_1(\beta_k r)\sin\beta_k z \qquad \begin{pmatrix}-b \leqslant z \leqslant 0\\ a \leqslant r < \infty\end{pmatrix} \qquad \left(\beta_k = \frac{(2k-1)\pi}{2b}\right)$$

$$\Psi_2(r, z) = \frac{1}{r}\int_0^\infty A(\xi)\,K_1(\xi r)\sin\xi z\,d\xi + \tag{1.4}$$

$$+\frac{1}{r}\int_0^\infty \xi B(\xi)\,e^{-z\xi}\,W_1(\xi r)\,d\xi \qquad \begin{pmatrix}0 \leqslant z < \infty\\ a \leqslant r < \infty\end{pmatrix}$$

$$\Psi_3(r, z) = \frac{1}{r}\int_0^\infty C(\xi)\,I_1(\xi r)\sin\xi z\,d\xi + \tag{1.5}$$

$$+\frac{1}{r}\sum_{k=1}^{\infty} D_k e^{-\lambda_k z} J_1(\lambda_k r) \qquad \begin{pmatrix}0 \leqslant z < \infty\\ 0 \leqslant r \leqslant a\end{pmatrix}$$

$$W_n(\xi r) = J_n(\xi r)\,Y_1(a\xi) - Y_n(\xi r)\,J_1(a\xi) \tag{1.6}$$

Here $J_n(x)$ and $Y_n(x)$ are Bessel functions of real argument, of the first and second kinds, respectively [8], $K_n(x)$ and $I_n(x)$ are modified Bessel functions of imaginary argument, and λ_k are roots of Eq. $J_1(\lambda a) = 0$.

Let us note that

$$W_1(\xi a) = 0, \qquad W_2(\xi a) = -W_0(\xi a) = \frac{2}{\pi a\xi} \tag{1.7}$$

We evaluate the displacements and stresses in the sub-domains D_i ($i = 1, 2, 3$) by means of Formulas

$$v^{(i)}(r,z) = r\Psi_i(r,z), \qquad \tau_{r\varphi}^{(i)} = G_i r\frac{\partial\Psi_i}{\partial r}, \tag{1.8}$$
$$\tau_{z\varphi}^{(i)} = G_i r\frac{\partial\Psi_i}{\partial z} \quad (i=1,2,3), \quad G_1 = G_2 = G$$

Utilizing (1.8), we obtain for (1.3) to (1.5)

$$\tau_{r\varphi}^{(1)}(r, z) = G\left\{-\int_0^\infty \xi^2[D(\xi)\operatorname{sh}\xi z + F(\xi)\operatorname{ch}\xi z]\,W_2(\xi r)\,d\xi - \right. \tag{1.9}$$
$$\left. -\sum_{k=1}^{\infty}\beta_k H_k K_2(\beta_k r)\sin\beta_k z\right\}$$

$$\tau_{z\varphi}^{(1)}(r, z) = G\left\{\int_0^\infty \xi^2[D(\xi)\operatorname{ch}\xi z + F(\xi)\operatorname{sh}\xi z]\,W_1(\xi r)\,d\xi + \right.$$
$$\left. +\sum_{k=1}^{\infty}\beta_k H_k K_1(\beta_k r)\cos\beta_k z\right\}$$

$$\tau_{r\varphi}^{(2)}(r, z) = G\left[-\int_0^\infty \xi A(\xi)\,K_2(\xi r)\sin\xi z\,d\xi - \int_0^\infty \xi^2 B(\xi)\,e^{-\xi z}W_2(\xi r)\,d\xi\right]$$

$$\tau_{z\varphi}^{(2)}(r, z) = G\left[\int_0^\infty \xi A(\xi)\,K_1(\xi r)\cos\xi z\,d\xi - \int_0^\infty \xi^2 B(\xi)\,e^{-\xi z}W_1(\xi r)\,d\xi\right]$$

$$\tau_{r\varphi}^{(3)}(r,\ z) = G_3\Big[\int_0^\infty \xi C(\xi) I_2(\xi r)\sin\xi z\, d\xi - \sum_{k=1}^\infty \lambda_k D_k e^{-\lambda_k z} J_2(\lambda_k r)\Big]$$

$$\tau_{z\varphi}^{(3)}(r,\ z) = G_3\Big[\int_0^\infty \xi C(\xi) I_1(\xi r)\cos\xi z\, d\xi - \sum_{k=1}^\infty \lambda_k D_k e^{-\lambda_k z} J_1(\lambda_k r)\Big]$$

The constant coefficients H_k and D_k and the functions $D(\xi)$, $F(\xi)$, $A(\xi)$, $B(\xi)$ and $C(\xi)$ should be determined from the following boundary conditions and connection conditions between the sub-domains D_i $(i = 1, 2, 3)$:

$$\tau_{z\varphi}^{(1)}(r,\ -b) = 0 \qquad (a \leqslant r < \infty)$$

$$v^{(1)}(a,\ z) = c \quad (-b \leqslant z \leqslant 0), \qquad v^{(3)}(r,\ 0) = \frac{c}{a} r \quad (0 \leqslant r \leqslant a)$$

$$v^{(3)}(a,\ z) = v^{(2)}(a,\ z), \quad \tau_{r\varphi}^{(3)}(a,\ z) = \tau_{r\varphi}^{(2)}(a,\ z) \qquad (0 \leqslant z < \infty) \tag{1.10}$$

$$v^{(2)}(r,\ 0) = v^{(1)}(r,\ 0), \quad \tau_{z\varphi}^{(2)}(r,\ 0) = \tau_{z\varphi}^{(1)}(r,\ 0) \qquad (a \leqslant z < \infty)$$

Utilizing (1.9) and satisfying (1.10), we obtain

$$D(\xi)\,\mathrm{ch}\,\xi b - F(\xi)\,\mathrm{sh}\,\xi b = 0, \quad F(\xi) = B(\xi) \qquad (m_0 = G/G_3)$$

$$C(\xi) I_1(\xi a) = A(\xi) K_1(\xi a)$$

$$H_k = -\frac{2c}{\beta_k b K_1(\beta_k a)}, \qquad D_k = \frac{2c}{\lambda_k a J_2(\lambda_k a)} \tag{1.11}$$

$$C(\xi) I_2(\xi a) + m_0 A(\xi) K_2(\xi a) = \frac{2c I_2(\xi a)}{\pi \xi I_1(\xi a)} - \frac{4m_0}{a\pi^2}\int_0^\infty \frac{tB(t)\,dt}{t^2+\xi^2}$$

$$\xi[D(\xi) + B(\xi)]\,[J_1^2(a\xi) + Y_1^2(a\xi)] = -\frac{2}{\pi}\int_0^\infty \frac{tA(t)K_1(ta)\,dt}{t^2+\xi^2} - \frac{c}{\xi}\,\mathrm{th}\, b\xi$$

Here we used the following Fourier and Weber-Orr integral transform formulas [8]:

$$\Phi(z) = \int_0^\infty f(\xi)\sin\xi z\, d\xi, \qquad f(\xi) = \frac{2}{\pi}\int_0^\infty \Phi(z)\sin\xi z\, dz \tag{1.12}$$

$$\varphi(r) = \int_0^\infty \xi\psi(\xi) W_1(\xi r)\, d\xi, \qquad \psi(\xi)\,[J_1^2(a\xi) + Y_1^2(a\xi)] = \int_0^\infty r\varphi(r) W_1(\xi r)\, dr$$

as well as the values of the integrals

$$\int_0^\infty e^{-tz}\sin\xi z\, dz = \frac{\xi}{t^2+\xi^2}, \quad \int_a^\infty r W_1(\xi r) K_1(\beta_k r)\, dr = -\frac{2K_1(\beta_k a)}{\pi(\xi^2+\beta_k^2)} \tag{1.13}$$

and the values of the series

$$\sum_{k=1}^\infty \frac{1}{\lambda_k^2+\xi^2} = \frac{a I_2(\xi a)}{2\xi I_1(\xi a)}, \qquad \sum_{k=1}^\infty \frac{1}{\beta_k^2+\xi^2} = \frac{b}{2\xi}\,\mathrm{th}\, b\xi \qquad \left(\beta_k = \frac{(2k-1)\pi}{2b}\right)$$

Summation is here over the roots of Eq. $J_1(\lambda a) = 0$. Equalities (1.7) are hence taken into account.

Eliminating $C(\xi)$ from the sixth Eq. of (1.11) and $D(\xi)$ from the last Eq. of (1.11), we obtain the system of integral Eqs.

$$A(\xi) = \frac{1}{\Omega(\xi a)}\left[\frac{2c I_2(\xi a)}{\pi\xi I} - \frac{4m_0 I_1(\xi a)}{a\pi^2}\int_0^\infty \frac{tB(t)\,dt}{t^2+\xi^2}\right] \tag{1.14}$$

$$\xi B(\xi) = -\frac{1}{(1+\mathrm{th}\, b\xi)\,[J_1^2(a\xi)+Y_1^2(a\xi)]}\left[\frac{c}{\xi}\,\mathrm{th}\, b\xi + \frac{2}{\pi}\int_0^\infty \frac{tA(t)K_1(ta)\,dt}{t^2+\xi^2}\right] \tag{1.15}$$

$$\Omega(\xi a) = I_2(\xi a) K_1(\xi a) + m_0 K_2(\xi a) I_1(\xi a) \tag{1.16}$$

Let us make the following change of unknown functions in (1.14) and (1.15):

$$tA(t)\ K_1(ta) = A^*(t), \qquad tB(t) = -B^*(t) \tag{1.17}$$

Then substituting (1.14) into (1.15), we obtain an integral Eq. to determine $B^*(t)$

$$B^*(\xi) = \int_0^\infty B^*(z) K(\xi, z)\, dz + F(\ ) \tag{1.18}$$

$$K(\xi, z) = \frac{8m_0}{a\pi^3 (1 + \operatorname{th} b\xi) [J_1^2(a\xi) + Y_1^2(a\xi)]} J^{(1)}(\xi, z) \tag{1.19}$$

$$F(\xi) = \frac{c}{(1 + \operatorname{th} b\xi) [J_1^2(a\xi) + Y_1^2(a\xi)]} \left[\frac{\operatorname{th} b\xi}{\xi} + \frac{4}{\pi^2} J^{(2)}(\xi, z)\right] \tag{1.20}$$

$$J^{(1)}(\xi z) = \int_0^\infty \frac{tI_1(ta) K_1(ta)\, dt}{\Omega(ta)(t^2 + \xi^2)(t^2 + z^2)}, \qquad J^{(2)}(\xi, z) = \int_0^\infty \frac{I_2(ta) K_1(ta)\, dt}{\Omega(ta)(t^2 + \xi^2)}$$

In order to show the solvability of the integral Eq. (1.18) and the possibility of utilizing successive approximations to construct the solution, let us estimate the value of the integral

$$\int_0^\infty |K(\xi, z)|\, dz \tag{1.21}$$

Using the inequality $K_n(x) \leqslant K_{n+1}(x)$, which is valid for Bessel functions of the second kind of imaginary argument, we will have

$$\frac{m_0 I_1(ta) K_1(ta)}{\Omega(ta)} \leqslant \frac{m_0 I_1(ta) K_2(ta)}{I_2(ta) K_1(ta) + m_0 K_2(ta) I_1(ta)} \leqslant 1 \tag{1.22}$$

Then we obtain the following estimate [9] for the integral in (1.19):

$$m_0 J^{(1)}(\xi, z) \leqslant \int_0^\infty \frac{t dt}{(t^2 + \xi^2)(t^2 + z^2)} = \frac{1}{z^2 - \xi^2} \ln \frac{z}{\xi} \tag{1.23}$$

Utilizing this estimate, we find

$$\int_0^\infty |K(\xi, z)|\, dz \leqslant \frac{8}{a\pi^3 (1 + \operatorname{th} \xi b) [J_1^2(a\xi) + Y_1^2(a\xi)]} \int_0^\infty \ln \frac{z}{\xi} \frac{dz}{z^2 - \xi^2} =$$

$$= \frac{2}{a\pi\xi (1 + \operatorname{th} \xi b) [J_1^2(a\xi) + Y_1^2(a\xi)]} = f(\xi) \tag{1.24}$$

Let us note that $f(\xi) \to 0$ as $\xi \to 0$; moreover, by virtue of the known equalities $xY_1(x)|_{x\to 0} = -1$ and $J_1(0) = 0$, we have

$$f(\xi)|_{\xi\to\infty} = 1/2 \tag{1.25}$$

if the following asymptotic formulas [8] are used for integer n as $z \to \infty$

$$J_n(z) \approx \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\pi n}{2} - \frac{\pi}{4}\right)$$

$$Y_n(z) \approx \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{\pi n}{2} - \frac{\pi}{4}\right) \tag{1.26}$$

For $\nu > 0$ and $x > 0$ the product $x\ [J_\nu^2(x) + Y_\nu^2(x)]$, considered as a function of x, will decrease monotonously [9 and 10] if $\nu > 1/2$.

Hence, $f(\xi)$ increase monotonously remaining always less than one-half.

Therefore, the inequality

$$\int_0^\infty |K(\xi, z)|\,dz \leqslant f(\xi) \leqslant \frac{1}{2} \qquad (0 \leqslant \xi < \infty) \tag{1.27}$$

holds.

It is easy to see that the right-hand side $F(\xi)$ of the integral Eq. (1.18) is also bounded. In fact

$$F(\xi) = \frac{a\pi}{2} f(\xi)\, c \operatorname{th} b\xi + \frac{2ac\xi}{\pi} f(\xi) \int_0^\infty \frac{I_2(ta) K_1(ta)\, dt}{\Omega(ta)(t^2+\xi^2)} \leqslant$$
$$\leqslant \frac{a\pi c}{2} f(\xi) \left[\operatorname{th} b\xi + \frac{4\xi}{\pi^2} \int_0^\infty \frac{dt}{t^2+\xi^2}\right] = \frac{a\pi c}{2} f(\xi) \left(\operatorname{th} b\xi + \frac{2}{\pi}\right) < \frac{a\pi c}{2} \tag{1.28}$$

The estimates (1.27) as well as the following relationships:

$$\int_0^\infty \frac{dt}{t^2+\xi^2} = \frac{\pi}{2\xi}, \qquad \frac{I_2(ta) K_1(ta)}{\Omega(ta)} < 1$$

have been used here.

But as is known [11], when conditions (1.27) and (1.28) are satisfied, the integral Eq. (1.18) is solvable in the class of functions bounded on the half-axis, and this solution may be constructed by successive approximations.

Determining the unknown function $B^*(\xi)$ from the fundamental integral Eq. (1.18), we first determine the function $A(\xi)$ by successive approximations from (1.14) and (1.17), and then the remaining unknown functions.

Using the solution obtained above, we consider some particular problems of the torsion of a half-space.

2. Torsion of a composite half-space with a vertical elastic core by the rotation of a stiff stamp attached to the core. In the particular case when $b = 0$ (Fig. 2), the solution of the problem of the torsion of a composite half-space with a vertical elastic core twisted by rotating a stiff stamp attached to the core, may be obtained from the general solution of the problem presented above.

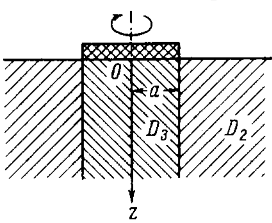

Fig. 2

In this case the boundary conditions of the problem become

$$v^{(3)}(r, 0) = cr / ar \quad (0 \leqslant r \leqslant a),$$
$$\tau_{z\varphi}^{(2)}(r, 0) = 0 \quad (a < r < \infty)$$
$$v^{(3)}(a, z) = v^{(2)}(a, z), \tag{2.1}$$
$$\tau_{r\varphi}^{(3)}(a, z) = \tau_{r\varphi}^{(2)}(a, z) \qquad (0 \leqslant z < \infty)$$

The solution of the problem may be represented as

$$\Psi_2(r, z) = \frac{1}{r} \int_0^\infty \frac{A^*(\xi) K_1(\xi r)}{\xi K_2(\xi a)} \sin \xi z\, d\xi - \frac{1}{r} \int_0^\infty B^*(\xi) e^{-\xi z} W_1(\xi r)\, d\xi \quad \begin{pmatrix} a \leqslant r < \infty \\ 0 \leqslant z < \infty \end{pmatrix} \tag{2.2}$$

$$\Psi_3(r, z) = \frac{1}{r} \int_0^\infty \frac{A^*(\xi) I_1(\xi r)}{\xi I_1(\xi a)} \sin \xi z\, d\xi + \frac{2c}{ar} \sum_{k=1}^\infty \frac{e^{-\lambda_k z} J_1(\lambda_k r)}{\lambda_k J_2(\lambda_k a)} \quad \begin{pmatrix} 0 \leqslant r \leqslant a \\ 0 \leqslant z < \infty \end{pmatrix} \tag{2.3}$$

The arbitrary integration functions $A^*(\xi)$ and $B^*(\xi)$ should be determined from the system of integral Eqs.

$$A^*(\xi) = \frac{4\xi m_0 I_1(\xi a) K_1(\xi a)}{a\pi^2 \Omega(\xi a)} \int_0^\infty \frac{B^*(t)\, dt}{t^2+\xi^2} + \frac{2c I_2(\xi a) K_1(\xi a)}{\pi \Omega(\xi a)} \tag{2.4}$$

$$B^*(\xi) = \frac{2}{\pi [J_1^2(a\xi) + Y_1^2(a\xi)]} \int_0^\infty \frac{A^*(t)\, dt}{t^2+\xi^2} \tag{2.5}$$

$$m_0 = \frac{G}{G_3}, \qquad \Omega(\xi a) = K_1(\xi a) I_2(\xi a) + m_0 I_1(\xi a) K_2(\xi a) \tag{2.6}$$

The system of integral equations (2.4) and (2.5) may be reduced to a Fredholm integral equation of the following form:

$$B^*(\xi) = \int_0^\infty K(\xi, u) B^*(u)\, du + F(\xi) \tag{2.7}$$

$$K(\xi, u) = \frac{8m_0}{a\pi^3 [J_1^2(a\xi) + Y_1^2(a\xi)]} \int_0^\infty \frac{t I_1(ta) K_1(ta)\, dt}{\Omega(ta)(t^2+\xi^2)(t^2+u^2)} \tag{2.8}$$

$$F(\xi) = \frac{4c}{\pi^2 [J_1^2(a\xi) + Y_1^2(a\xi)]} \int_0^\infty \frac{I_2(ta) K_1(ta)\, dt}{\Omega(ta)(t^2+\xi^2)} \tag{2.9}$$

The estimate (1.27) holds for this equation, and therefore, its solution may be constructed by using successive approximations.

In conclusion, let us note that if a stiff core replaces the elastic core in the considered problem, the solution is obtained by elementary means with the aid of the displacement function $\Psi = ac/r^2$, where a is the core radius, and c is the angle of rotation. However, its solution is meaningless since the torque is infinite.

3. Torsion of an elastic half-space with a vertical cylindrical hole by rotating a stiff finite cylinder welded into the upper part of the hole. From the solution obtained in the first section we can also obtain the solution for the problem of torsion of a half-space with a vertical cylindrical hole twisted by means of the rotation of a stiff finite cylinder welded into the upper part of this hole (Fig. 3).

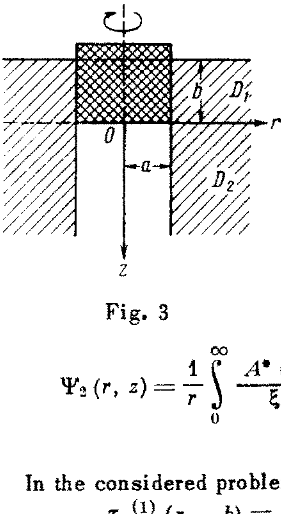

Fig. 3

Putting $m_0 = \infty$ here, we obtain the following expressions for the displacement functions in (1.3) to (1.5):

$$\Psi_1(r, z) = -\frac{1}{r}\int_0^\infty B^*(\xi) \frac{\operatorname{ch}(z+b)\xi}{\operatorname{ch}\xi b} W_1(\xi r)\, d\xi - \frac{2c}{br}\sum_{k=1}^\infty \frac{K_1(\beta_k r)\sin\beta_k z}{\beta_k K_1(\beta_k a)} \tag{3.1}$$

$$(-b \leqslant z \leqslant 0, \quad a \leqslant r < \infty)$$

$$\Psi_2(r, z) = \frac{1}{r}\int_0^\infty \frac{A^*(\xi) K_1(\xi r)}{\xi K_1(\xi a)} \sin \xi z\, d\xi - \frac{1}{r}\int_0^\infty B^*(\xi) e^{-z\xi} W_1(\xi r)\, d\xi \tag{3.2}$$

$$(0 \leqslant z < \infty, \quad a \leqslant r < \infty)$$

In the considered problem the boundary conditions become

$$\begin{gathered}\tau_{z\varphi}^{(1)}(r, -b) = 0 \quad (a \leqslant r < \infty), \qquad v^{(1)}(a, z) = c \quad (-b \leqslant z \leqslant 0)\\ \tau_{\varphi r}^{(2)}(a, z) = 0 \quad (0 \leqslant z < \infty)\\ v^{(1)}(r, 0) = v^{(2)}(r, 0), \qquad \tau_{z\varphi}^{(1)}(r, 0) = \tau_{z\varphi}^{(2)}(r, 0) \quad (a \leqslant r < \infty)\end{gathered} \tag{3.3}$$

Substituting (3.1) and (3.2) into (1.10) and taking account of condition (3.3) here, we obtain the following system for the unknown functions $A^*(\xi)$ and $B^*(\xi)$:

$$A^*(\xi) = \frac{4\xi K_1(\xi a)}{a\pi^2 K_2(\xi a)} \int_0^\infty \frac{B^*(t)\, dt}{t^2+\xi^2} \tag{3.4}$$

$$B^*(\xi) = \frac{1}{(1+\operatorname{th} b\xi)[J_1^2(a\xi) + Y_1^2(a\xi)]} \left[\frac{c}{\xi}\operatorname{th} b\xi + \frac{2}{\pi}\int_0^\infty \frac{A^*(t)\, dt}{t^2+\xi^2}\right] \tag{3.5}$$

This system of integral Eqs. may be reduced to a Fredholm integral equation of the second kind of the following form:

$$B^*(\xi) = \int_0^\infty K(\xi, u) B^*(u)\, du + F(\xi) \tag{3.6}$$

$$K(\xi, u) = \frac{8}{a\pi^3 (1 + \operatorname{th} b\xi)[J_1^2(a\xi) + Y_1^2(a\xi)]} \int_0^\infty \frac{tK_1(ta)\, dt}{K_2(ta)(t^2 + \xi^2)(t^2 + u^2)}$$

$$F(\xi) = \frac{c \operatorname{th} b\xi}{\xi (1 + \operatorname{th} b\xi)[J_1^2(a\xi) + Y_1^2(a\xi)]}$$

The estimate (1.27) is also valid for this equation, and therefore, its solution may be constructed by successive approximations.

Therefore, by having the value of $B^*(\xi)$ and then determining the value of $A^*(\xi)$ by using (3.4), we find the stress and displacement components.

BIBLIOGRAPHY

1. Rostovtsev, N.A., On the problem of the torsion of an elastic half-space. PMM, Vol. 19, No. 1, 1955.
2. Reissner, E. and Sagoci, H., Forced torsional oscillations of an elastic half-space. J. Appl. Phys. Vol. 15, No. 9, 1944.
3. Ufliand, IA.S., Torsion of an elastic layer. Dokl. Akad. Nauk SSSR, Vol. 129, No. 5, 1959.
4. Florence, A.L., Two contact problems for an elastic layer. Quart. J. Mech. and Appl. Math., Vol. 14, part 4, 1961.
5. Grilitskii, D.V., Torsion of a two-layered elastic medium. Prikl. Mekh. Vol. 7, No. 1, 1961.
6. Grilitskii, D.V. and Kizyma, IA.M., Combined torsion of a rod and half-space Prikl. Mekh. Vol. 3, No. 2, 1967.
7. Arutiunian, N.KH. and Babloian, A.A., Contact problems for a half-space with an inclusion. PMM, Vol. 30, No. 6, 1966.
8. Bateman, H. and Erdelyi, A., Higher Transcendental Functions. Vol. 2, Moscow, "Nauka", 1966.
9. Gradshtein, I.S. and Ryzhik, I.M., Tables of Integrals, Sums, Series, and Products. 4th Ed. Moscow, Fizmatgiz, 1962.
10. Magnus, W. and Oberhettinger, F., Formeln und Sätze für die speziellen Funktionen der mathematischen Physik. Berlin, Springer-Verlag, 1948.
11. Liusternik, L.A. and Sobolev, V.I., Elements of Functional Analysis 2nd Ed., Moscow, "Nauka", 1965.

Translated by M.D.F.